# Kaluza Klein Theory

## A journey into the fifth dimension

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- A unification of general relativity and electromagnetism.
- Inspiration for many higher dimensional unification theories (such as string theory).
- Some history:
  - 1921: Kaluza unified GR and EM with a fifth dimension.
  - 1926: Klein developed the theory further. He argued that the fifth dimension is small and compact.

# An observation

- Gauge transformation in EM,

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \Lambda, \tag{1}$$

leaves the electromagnetic field tensor $F_{\mu\nu}$ invariant.

- A coordinate transformation

$$x^\mu \to y^\mu = x^\mu + \epsilon^\mu(x)$$

causes the metric in linearized gravity to change by

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu \tag{2}$$

(for infinitesimal $\epsilon$)

- Note the similarity between 1 and 2. Can we find a way to make 1 be caused by a coordinate transformation?

# A new dimension

- Add a fifth dimension and promote $h_{\mu\nu}$ to $h_{MN}$ where $M, N \in \{0, 1, 2, 3, 4\}$.
- We can identify

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \Lambda, \tag{3}$$

with

$$h_{\mu 4} \to h'_{\mu 4} = h_{\mu 4} - \partial_\mu \epsilon_4 - \partial_4 \epsilon_\mu \tag{4}$$

if

$$h_{\mu 4} = lA_\mu, \tag{5}$$

$$\epsilon_4 = l\Lambda, \tag{6}$$

$$\partial_4 \epsilon_\mu = 0. \tag{7}$$

- We now have that the EM gauge transformation is just a change in the five-dimensional metric under a change of coordinates

$$y^\mu = x^\mu \tag{8}$$

$$y^4 = x^4 + l\Lambda(x^0, x^1, x^2, x^3) \tag{9}$$

# Finding the general metric

Consider how $dx^4$ change under $y^4 = x^4 + l\Lambda(x^\mu)$, $y^\mu = x^\mu$:

$$dx^4 = \frac{\partial x^4}{\partial y^{M'}} dy^{M'} = (\delta^4_{M'} - l\partial_{M'}\Lambda)dy^{M'}$$
$$= dy^4 - l\partial_{\mu'}\Lambda dy^{\mu'}$$

We can create now the combination $(dx^4 + A_\mu dx^\mu)$, which will transform as

$$dx^4 + lA_\mu dx^\mu = dy^4 - l\partial_{\mu'}\Lambda dy^{\mu'} + lA_\mu dy^\mu$$
$$= dy^4 + l(A_{\mu'} - \partial_{\mu'}\Lambda)dy^{\mu'}$$

Using this term, we can finally write down the line element

## Kaluza-Klein line element

$$ds^2 = g_{\mu\nu}dx^\nu dx^\nu + (dx^4 + lA_\mu dx^\mu)^2 \tag{10}$$

# A closer look at the fifth dimension

- By including another dimension, two problems immediately arise:
  - How come we don't see the extra dimension?
  - Why don't we see any particles disappearing into the fifth dimension?
- Klein's solution: the fifth dimension is very small and compact! We identify $x^4 \sim x^4 + 2\pi a$.
  - If $a$ is very small, we might mistake the small circles for points.
  - If a particle were to escape into the fifth dimension, its wave function would have to be confined in an area less than $2\pi a$, which would require an extreme energy.

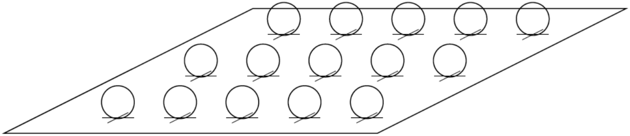

Figure: Illustration from Ø. Grøn, S. Hervik (2007)

# The action

## The metric

$$G_{MN} = \begin{bmatrix} g_{\mu\nu} + l^2 A_\mu A_\nu & lA_\mu \\ lA_\mu & 1 \end{bmatrix} \quad (11)$$

- We can now write down the action exactly as in GR,

$$S = \int d^5 x M_{KK}^3 \sqrt{-G} R(G), \quad (12)$$

where $M_{KK}$ is needed to make $S$ dimensionless. It plays a role similar to the Planck mass.

# Reproducing EM and GR

- We consider two scenarios which should reproduce GR and EM respectively.

## Scenario 1

- $A_\mu = 0 \implies S = 2\pi a M_{KK}^3 \int d^4x \sqrt{-g} R(g)$
- This should be equal to the Einstein-Hilbert action , which requires

$$2\pi a M_{KK}^3 = M_P^2$$

## Scenario 2

- $g_{\mu\nu} = \eta_{\mu\nu} \implies S = -\frac{1}{4}(M_P l)^2 \int d^4x F_{\mu\nu} F^{\mu\nu}$
- To match EM we need

$$M_p l = 1 \implies l = l_p \tag{13}$$

# Motion in the fifth dimension

- Until now we had no way to fix $a$. If we vary the action

$$S_{particle} = -m \int d\tau = -m \int \left[-\eta_{\mu\nu} dx^{\mu} dx^{\nu} + (dy + lA_{\mu} dx^{\mu})^2\right]^{\frac{1}{2}} \tag{14}$$

we get

$$p \equiv m \left( \frac{dx^4}{d\tau} + lA_{\mu} \frac{dx^{\mu}}{d\tau} \right) = \text{conserved} \tag{15}$$

$$m \frac{d^2 x^{\mu}}{d\tau^2} = (pl) F^{\mu}_{\nu} \frac{dx^{\nu}}{d\tau} \tag{16}$$

- This is identical to the equation of motion of a charged particle in an electromagnetic field if $q = pl$. Hence, we see that charge corresponds to a motion in the fifth dimension!

# Quantizing the charge

- In quantum mechanics, the wave function of a particle on a circle with constant potential is

$$\exp\left[\frac{ipx^4}{\hbar}\right]. \tag{17}$$

Requiring $\psi(x^4) = \psi(x^4 + 2\pi a)$ gives

$$p = \frac{n\hbar}{a} \implies q_n = \frac{n\hbar l}{a} \tag{18}$$

- Charge is quantized!
- Requiring $q_1 = e$ gives

$$a \sim 3l_p \tag{19}$$

- Indeed, the dimension is very small!

# Final remarks

- Good things
  - Unification of GR and EM.
  - Quantization of charge follows.
  - Antimatter emerges naturally when $p^4 \to -p^4$.
  - CPT symmetry now deals exclusively with space-time concepts.
  - Additional dimensions can be added to include non-abelian gauge theories.
- Bad things
  - The theory cannot distinguish between left- and right-handedness.
  - Fermions does not arise naturally.

# The end

Thank you for your attention!

# For Further Reading I


A. Zee.
*Einstein Gravity in a Nutshell*.
Princeton University Press, 2013.

Ø. Grøn, S. Hervik.
*Einstein's general theory of relativity : with modern applications in cosmology*.
Springer, 2007.